ALTERNATIVE SOLUTIONS OF INHOMOGENEOUS SECOND–ORDER LINEAR DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT. We exhibit an alternative method for solving inhomogeneous second–order linear ordinary dynamic equations on time scales, based on reduction of order rather than variation of parameters. Our form extends recent (and long-standing) analysis on $\mathbb{R}$ to a new form for difference equations, quantum equations, and arbitrary dynamic equations on time scales.

1. INTRODUCTION TO SECOND-ORDER ORDINARY DYNAMIC EQUATIONS

A very common equation in mathematics, mathematical physics, and engineering is the inhomogeneous second–order linear ordinary differential equation

$$(1.1) \qquad y''(t) + p(t)y'(t) + q(t)y(t) = r(t), \quad t \in \mathbb{R},$$

its ordinary difference equation counterparts

$$(1.2) \qquad \Delta(\Delta y)(t) + p(t)\Delta y(t) + q(t)y(t) = r(t), \quad t \in \mathbb{Z}, \quad \Delta y(t) := y(t+1) - y(t)$$

or

$$(1.3) \qquad y(t+2) + \alpha(t)y(t+1) + \beta(t)y(t) = r(t), \quad t \in \mathbb{Z},$$

the related ordinary quantum equation

$$(1.4) \qquad D_h(D_h y)(t) + p(t)D_h y(t) + q(t)y(t) = r(t), \quad t \in h^{\mathbb{Z}}, \quad h > 1, \quad D_h y(t) := \frac{y(ht) - y(t)}{ht - t},$$

or the recently introduced ordinary dynamic equation on time scales given by

$$(1.5) \qquad y^{\Delta\Delta}(t) + p(t)y^{\Delta}(t) + q(t)y(t) = r(t), \quad t \in \mathbb{T}^{\kappa^2},$$

where the differential/shift operators in $(1.1)-(1.5)$ represent differentiation with respect to t on the corresponding time scales, respectively. In the general setting represented by (1.5), the functions p, q, and r are real-valued right-dense continuous scalar functions of t satisfying the regressivity condition

$$1 + \mu(t)\left[-p(t) + \mu(t)q(t)\right] \neq 0.$$

Recall that on a time scale $\mathbb{T}$, namely any nonempty closed subset of the real line, the delta derivative is given by

$$y^{\Delta}(t) := \lim_{s \to t} \frac{y^{\sigma}(t) - y(s)}{\sigma(t) - s}, \quad t \in \mathbb{T}^{\kappa},$$

provided the limit exists, where σ is the forward jump operator $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, and $y^{\sigma} = y \circ \sigma$. The graininess μ is simply $\mu(t) = \sigma(t) - t$. For more on time scales see Hilger [7].

Indeed, extensive analysis of (1.5) and its solution forms can be found in Bohner and Peterson [3], while the ordinary differential equation (1.1) is studied by, for example, Blest [2], Boyce and DiPrima [4], Hille [8], Ince [9], Johnson, Busawon, and Barbot [11], Kelley and Peterson [13], whereas the difference equation (1.2) appears in Agarwal [1], Elaydi [6], and Kelley and Peterson [12]. A commonly used technique to solve $(1.1)-(1.5)$ is Lagrange's variation of parameters method. In this approach, a solution y of $(1.1)-(1.5)$ takes the form $y = y_u + y_d$, where y_u is the complementary solution of the corresponding homogeneous or undriven ($r \equiv 0$) form of $(1.1)-(1.5)$, and y_d is any particular solution of the inhomogeneous or driven equations $(1.1)-(1.5)$. If y_1 and y_2 are two linearly independent solutions of the corresponding homogeneous equation, then it is well known that we may write $y = c_1 y_1 + c_2 y_2 + y_d$ for arbitrary constants c_1 and c_2. For example, using variation of parameters, the form of a particular solution for (1.5) is [3, Theorem 3.73]

$$(1.6) \qquad y_d(t) = y_2(t) \int_a^t \frac{y_1^{\sigma}(s)r(s)}{W^{\sigma}(y_1, y_2)(s)} \Delta s - y_1(t) \int_a^t \frac{y_2^{\sigma}(s)r(s)}{W^{\sigma}(y_1, y_2)(s)} \Delta s, \quad t \in \mathbb{T},$$

which reduces to

$$y_d(t) = y_2(t) \int_a^t \frac{y_1(s)r(s)}{W(y_1, y_2)(s)} ds - y_1(t) \int_a^t \frac{y_2(s)r(s)}{W(y_1, y_2)(s)} ds, \quad t \in \mathbb{R}$$

for (1.1), and to

$$y_d(t) = y_2(t) \sum_{s=a}^{t-1} \frac{y_1(s+1)r(s)}{W(y_1, y_2)(s+1)} - y_1(t) \sum_{s=a}^{t-1} \frac{y_2(s+1)r(s)}{W(y_1, y_2)(s+1)}, \quad t \in \mathbb{Z}$$

for (1.2), where in each case $W(y_1, y_2)$ is the Wronskian of y_1 and y_2, defined appropriately for each time scale. Notice that the integrals/summations always involve both y_1 and y_2. Recently, Johnson, Busawon, and Barbot [11] derived two alternative solution forms for y_d for the ordinary differential equation (1.1), namely

$$(1.7) \qquad y_d(t) = y_i(t) \int \frac{e^{-\int p(t)dt} \int r(t)y_i(t)e^{\int p(t)dt}dt}{y_i^2(t)} dt, \quad t \in \mathbb{R}, \quad i = 1, 2.$$

As the authors point out in [11], either y_1 or y_2 may be chosen in (1.7), depending on which one yields an easier integral to compute.

Unfortunately neither the technique nor the results in [11] are new, as they are discussed in Blest [2] and the undergraduate textbook by Boyce and DiPrima [4, Exercise 28, p185], to cite just two examples. In fact, Yosida [14, p.29] calls the technique D'Alembert's reduction of order for ordinary differential equations, while according to Jahnke [10, p.332], the method of reduction of order dates back at least as far as Euler (1750). In addition, Clairaut, Lagrange and Laplace have been involved with the method and thus it may be impossible to attribute to one mathematician [10, p.332]. In any case, the method is at least 250 years old.

Despite the non-novelty of (1.7) for the ode (1.1), such a result would be new on general time scales for (1.5). Consequently one of our goals in this paper is to generalize and extend (1.7) to an alternative form for particular solutions to the ordinary dynamic equation on time scales (1.5), which would then nevertheless result in new alternative forms for the difference equations (1.2) or (1.3), and the quantum equation (1.4) as simple corollaries, as well as contain (1.7) for (1.1). This development is different from that given by Bohner and Peterson [3, Chapter 3].

An additional goal of this paper is to present some simple results that guarantee the nonmultiplicity of solutions to initial value problems associated with (1.5). Our approach is based on simple inequalities and does not require a knowledge of matrix theory, in contrast to [3, Theorem 3.1; Corollary 5.90].

In Section 2 we address the question of nonmultiplicity of solutions, while in Section 3 we develop a method for solving inhomogeneous second–order linear ordinary dynamic equations on time scales, based on reduction of order rather than variation of parameters. Section 4 contains some special cases that illustrate our results.

2. NONMULTIPLICITY

In this section we consider the notion of nonmultiplicity of solutions to the linear initial value problem (1.5) with initial conditions

$$(2.1) \qquad y(t_0) = A, \qquad y^\Delta(t_0) = B,$$

where $A, B \in \mathbb{R}$ and $t_0 \in \mathbb{T}$. Let $I \subseteq \mathbb{R}$ and $t_0 \in I_\mathbb{T} := I \cap \mathbb{T}$. By "nonmultiplicity of solutions" we mean that our theorems will present conditions under which (1.5), (2.1) will have, at most, one solution $y = y(t)$ for $t \geq t_0$, $t \in I_\mathbb{T}$. Such information is highly valuable, for example, when constructing explicit solutions to problems as we can determine when the constructed solution (or unique linear combination of solutions) will be the only solution to the problem at hand. Our techniques follow those of Coddington [5, Ch.2, Sec.3].

In the following, we will make use of the time-scale exponential function defined in terms of right-dense continuous functions p satisfying the regressivity condition $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$. Given such a p, the delta exponential function [3, Theorem 2.30] is given by

$$e_p(t, a) = \begin{cases} \exp\left(\int_a^t p(\tau)\Delta\tau\right) & : \mu(\tau) = 0, \\ \exp\left(\int_a^t \frac{1}{\mu(\tau)} \text{Log}(1 + p(\tau)\mu(\tau))\Delta\tau\right) & : \mu(\tau) \neq 0, \end{cases}$$

where Log is the principal logarithm. It follows that $e_p(t, a)$ is the unique solution to the initial value problem $\phi^\Delta(t) = p(t)\phi(t)$, $\phi(a) = 1$ on $\mathbb{T}$. We will denote $1/e_p(t, a)$ by $e_{\ominus p}(t, a)$.

We will require the following lemma from [3, Theorem 6.1, p.255].

Lemma 2.1. *Let $\ell \in \mathcal{R}^+$ and $v \in C^1_{\text{rd}}(\mathbb{T})$, and let $t_0 \in \mathbb{T}$. If $v^\Delta(t) \leq \ell(t)v(t)$ for all $t \geq t_0$, then $v(t) \leq v(t_0)e_\ell(t, t_0)$ for all $t \geq t_0$.*

Our initial analysis will concern the case of (1.5) with constant coefficients, namely

(2.2) $$y^{\Delta\Delta}(t) + py^{\Delta}(t) + qy(t) = r(t), \quad y(t_0) = A, \quad y^{\Delta}(t_0) = B,$$

where $p, q \in \mathbb{R}$ are constants. The following result involves the homogeneous form of (2.2), that is

(2.3) $$y^{\Delta\Delta}(t) + py^{\Delta}(t) + qy(t) = 0,$$

and provides an estimate on the growth rate of solutions to (2.3). For this estimate, we define

$$\|y(t)\|_2 := \left((y(t))^2 + \left(y^{\Delta}(t) \right)^2 \right)^{1/2}, \quad t \in I_{\mathbb{T}}, \quad t \geq t_0.$$

Theorem 2.2. *Let $k := 1 + |p| + |q|$. If y is any solution to (2.3) on $I_{\mathbb{T}}$ then*

(2.4) $$\|y(t)\|_2 \leq \|y(t_0)\|_2 e_k(t, t_0), \quad t \in I_{\mathbb{T}}, \quad t \geq t_0.$$

Proof. Let $u(t) = \|y(t)\|_2^2$, where y is a solution to (2.3). For all $t \in I_{\mathbb{T}}^{\kappa}$ we have

$$
\begin{aligned}
u^{\Delta}(t) &= (y(t) + y^{\sigma}(t)) \, y^{\Delta}(t) + \left(y^{\Delta}(t) + y^{\Delta\sigma}(t) \right) y^{\Delta\Delta}(t) \\
&= \left(2y(t) + \mu(t)y^{\Delta}(t) \right) y^{\Delta}(t) + \left(2y^{\Delta}(t) + \mu(t)y^{\Delta\Delta}(t) \right) y^{\Delta\Delta}(t) \\
&= 2y(t)y^{\Delta}(t) + \mu(t)(y^{\Delta}(t))^2 + 2y^{\Delta}(t)y^{\Delta\Delta}(t) + \mu(t)(y^{\Delta\Delta}(t))^2.
\end{aligned}
$$

Now, apply Young's inequality $2ab \leq a^2 + b^2$ to the first term and replace $y^{\Delta\Delta}$ with $-py^{\Delta} - qy$ to obtain

$$
\begin{aligned}
u^{\Delta}(t) &\leq (y(t))^2 + (y^{\Delta}(t))^2 + \mu(t)(y^{\Delta}(t))^2 + 2y^{\Delta}(t)[-py^{\Delta}(t) - qy(t)] + \mu(t)[-py^{\Delta}(t) - qy(t)]^2 \\
&\leq (1 + 2|p| + |q|) \left((y(t))^2 + (y^{\Delta}(t))^2 \right) + \mu(t) \left[1 + p^2 + |p||q| + q^2 \right] \left((y(t))^2 + (y^{\Delta}(t))^2 \right) \\
&\leq 2(1 + |p| + |q|) \left((y(t))^2 + (y^{\Delta}(t))^2 \right) + \mu(t) \left[1 + 2(|p| + |q|) + p^2 + 2|p||q| + q^2 \right] \left((y(t))^2 + (y^{\Delta}(t))^2 \right) \\
&= (k \oplus k)u(t).
\end{aligned}
$$

Above, $\oplus$ is known as the "circle plus" operator, $z \oplus w := z + w + \mu zw$, see [3, p.54].

Thus, the conditions of Lemma 2.1 hold with $v = u$ and $\ell = k \oplus k$. Consequently,

$$u(t) \leq u(t_0)e_{k \oplus k}(t, t_0) = u(t_0)(e_k(t, t_0))^2, \quad t \geq t_0, \quad t \in I_{\mathbb{T}},$$

and therefore

$$\|y(t)\|_2 \leq \|y(t_0)\|_2 e_k(t, t_0), \quad t \geq t_0, \quad t \in I_{\mathbb{T}}.$$

$\square$

Theorem 2.2 now leads to the following nonmultiplicity result for solutions to (2.2), (2.1).

Theorem 2.3. *Let $r \in \mathrm{C_{rd}}(I_{\mathbb{T}}^{\kappa^2}; \mathbb{R})$, and $t_0 \in I_{\mathbb{T}}$. The dynamic initial value problem (2.2), (2.1) has, at most, one solution $y = y(t)$ for $t \geq t_0$ with $t \in I_{\mathbb{T}}$.*

Proof. Assume there are two solutions x and y. Let $z(t) := x(t) - y(t)$ for $t \in I_{\mathbb{T}}$, and note that z must satisfy the homogeneous equation (2.3) together with the homogeneous initial conditions $z(t_0) = 0$, $z^{\Delta}(t_0) = 0$. Now, by Theorem 2.2 we have

$$\|z(t)\|_2 \leq \|z(t_0)\|_2 e_k(t, t_0) = 0, \quad t \geq t_0, \quad t \in I_{\mathbb{T}},$$

and thus $z(t) = 0$ for all $t \geq t_0$, $t \in I_{\mathbb{T}}$. It follows that $x = y$. □

The following result is an extension of Theorem 2.2 and concerns estimates on solutions to the homogeneous form of (1.5) with variable coefficients, namely

$$(2.5) \qquad y^{\Delta\Delta}(t) + p(t)y^{\Delta}(t) + q(t)y(t) = 0, \quad t \in \mathbb{T}^{\kappa^2}.$$

Theorem 2.4. *Let $t \in I_{\mathbb{T}}$. Let $p_1, q_1 \in \mathbb{R}$ be nonnegative constants such that*

$$|p(t)| \leq p_1, \quad |q(t)| \leq q_1, \quad t \in I_{\mathbb{T}}^{\kappa^2}, \quad t \geq t_0,$$

and let $k_1 := 1 + p_1 + q_1$. If y is any solution to (2.5) on $I_{\mathbb{T}}$, then

$$\|y(t)\|_2 \leq \|y(t_0)\|_2 e_{k_1}(t, t_0), \quad t \geq t_0, \quad t \in I_{\mathbb{T}}.$$

Proof. Our proof follows similar lines as that of the proof of Theorem 2.2, and thus we just sketch the details. Letting $u(t) = \|y(t)\|_2^2$, we obtain

$$
\begin{aligned}
u^{\Delta}(t) \ &\leq \ (y(t))^2 + (y^{\Delta}(t))^2 + \mu(t)(y^{\Delta}(t))^2 + 2y^{\Delta}(t)[-p(t)y^{\Delta}(t) - q(t)y(t)] + \mu(t)[-p(t)y^{\Delta}(t) - q(t)y(t)]^2 \\
&\leq \ 2(1 + p_1 + q_1)u(t) + \mu(t)\left[1 + 2(p_1 + q_1) + p_1^2 + 2p_1 q_1 + q_1^2\right]u(t) \\
&\leq \ (k_1 \oplus k_1)u(t).
\end{aligned}
$$

Thus, applying Lemma 2.1 we obtain

$$u(t) \leq u(t_0)(e_{k_1}(t, t_0))^2, \quad t \geq t_0, \quad t \in I_{\mathbb{T}},$$

and therefore

$$\|y(t)\|_2 \leq \|y(t_0)\|_2 e_{k_1}(t, t_0), \quad t \geq t_0, \quad t \in I_{\mathbb{T}}.$$

□

Theorem 2.4 now leads to the following nonmultiplicity result for solutions to (1.5), (2.1).

Theorem 2.5. *Let $t \in I_{\mathbb{T}}$. The dynamic equation (1.5) with initial conditions (2.1) has at most one solution $y = y(t)$ for $t \geq t_0$ with $t \in I_{\mathbb{T}}$.*

Proof. Assume there are two solutions x and y, and let $z(t) = x(t) - y(t)$ for $t \geq t_0$ with $t \in I_{\mathbb{T}}$. Note that z satisfies (2.5) and the initial homogeneous initial conditions $z(t_0) = 0$, $z^{\Delta}(t_0) = 0$. Since $I_{\mathbb{T}}$ may be unbounded, the coefficient functions p and q may not be bounded on $I_{\mathbb{T}}$, and so Theorem 2.4 may not be directly applied to z. We let t be any point in $I_{\mathbb{T}}$ such that $t > t_0$, and let $J_{\mathbb{T}}$ be any closed, bounded interval of $I_{\mathbb{T}}$ such that $J_{\mathbb{T}}$ has t_0 as a

left endpoint and $J_\mathbb{T}$ contains t. On $J_\mathbb{T}$, p and q are both bounded, say by p_1 and q_1, respectively. We can now apply Theorem 2.4 to z on $J_\mathbb{T}$, and so $z(t) = 0$ for all $t \in J_\mathbb{T}$. Now since t was chosen to be any point in $I_\mathbb{T}$ with $t > t_0$, we have shown that $x(t) = y(t)$ for all $t \in I_\mathbb{T}$ with $t \geq t_0$.   □

Remark 2.6. The quest for nonmultiplicity of solutions on intervals to the left of t_0 is a more delicate affair. The results of this section may be extended to include nonmultiplicity of solution for $t \leq t_0$ by adapting the proofs and obtaining inequalities like

$$u^\Delta(t) \geq -2ku(t), \quad \text{for all} \quad t \leq t_0, \ t \in I_\mathbb{T}.$$

However, there is a price to pay – the graininess function of the time scale would need to be bounded above. This is due to regressivity coming into play.

3. ALTERNATIVE SOLUTION FORMS FOR THE GENERAL INHOMOGENEOUS DYNAMIC EQUATION

In this section we state and prove the main result, namely a new form for a solution of (1.5) in the spirit of (1.7). As mentioned previously, this will then give us a new form for the difference equation (1.2) and quantum equation (1.4) as well.

Theorem 3.1. *Let $\mathbb{T}$ be a time scale and let p, q and r be real–valued right–dense continuous scalar functions of $t \in \mathbb{T}$ with p and q satisfying the regressivity condition*

$$(3.1) \qquad\qquad\qquad 1 + \mu(t)\left[-p(t) + \mu(t)q(t)\right] \neq 0, \quad t \in \mathbb{T}^\kappa.$$

For all $t \in \mathbb{T}$, let y_1 and y_2 satisfy

$$(3.2) \qquad\qquad\qquad y_i^{\Delta\Delta}(t) + p(t)y_i^\Delta(t) + q(t)y_i(t) = 0, \quad i = 1, 2,$$

and $W(t) := y_1(t)y_2^\Delta(t) - y_2(t)y_1^\Delta(t) \neq 0$. The general solution of the linear inhomogeneous second–order ordinary dynamic equation

$$y^{\Delta\Delta}(t) + p(t)y^\Delta(t) + q(t)y(t) = r(t), \quad t \in \mathbb{T}^{\kappa^2},$$

is

$$(3.3) \qquad y(t) = c_1 y_1(t) + c_2 y_2(t) + y_1(t) \int \frac{e_{(-p+\mu q)}(t, a) \int r(t)y_1^\sigma(t)e_{\ominus(-p+\mu q)}(\sigma(t), a)\Delta t}{y_1(t)y_1^\sigma(t)} \Delta t$$

or

$$(3.4) \qquad y(t) = c_1 y_1(t) + c_2 y_2(t) + y_2(t) \int \frac{e_{(-p+\mu q)}(t, a) \int r(t)y_2^\sigma(t)e_{\ominus(-p+\mu q)}(\sigma(t), a)\Delta t}{y_2(t)y_2^\sigma(t)} \Delta t,$$

where c_1 and c_2 are arbitrary constants and $e_{(-p+\mu q)}(\cdot, a)$ is the time–scale exponential function.

Proof. Clearly (3.3) and (3.4) are of the expected form $y = c_1 y_1 + c_2 y_2 + y_d$. One could easily verify that a function of the form

$$(3.5) \qquad\qquad y_d(t) = y_i(t) \int \frac{e_{(-p+\mu q)}(t, a) \int r(t)y_i^\sigma(t)e_{\ominus(-p+\mu q)}(\sigma(t), a)\Delta t}{y_i(t)y_i^\sigma(t)} \Delta t$$

is a particular solution of (1.5) using the time scale calculus, but that would not give any insight into where (3.5) comes from. Thus, to derive (3.5), assume we have a particular solution to the inhomogeneous equation (1.5) of the form $y_d(t) = y_i(t)v(t)$, where y_i solves the homogeneous equation (3.2) and v is a function to be determined. Then, using the product rule $(fg)^\Delta = gf^\Delta + f^\sigma g^\Delta$, we have

$$
\begin{aligned}
y_d^\Delta(t) &= v(t)y_i^\Delta(t) + y_i^\sigma(t)v^\Delta(t) \\
&= v(t)y_i^\Delta(t) + y_i(t)y_i^\sigma(t)v^\Delta(t)/y_i(t),
\end{aligned}
$$

and using the product rule again with the quotient rule $\left(\frac{f}{g}\right)^\Delta = \frac{gf^\Delta - fg^\Delta}{gg^\sigma}$ we see that

$$
\begin{aligned}
y_d^{\Delta\Delta}(t) &= v(t)y_i^{\Delta\Delta}(t) + y_i^{\Delta\sigma}(t)v^\Delta(t) + \frac{y_i(t)\left(y_i(t)y_i^\sigma(t)v^\Delta(t)\right)^\Delta - y_i(t)y_i^\sigma(t)v^\Delta(t)y_i^\Delta(t)}{y_i(t)y_i^\sigma(t)} \\
&= v(t)y_i^{\Delta\Delta}(t) + y_i^{\Delta\sigma}(t)v^\Delta(t) + \left(y_i(t)y_i^\sigma(t)v^\Delta(t)\right)^\Delta/y_i^\sigma(t) - v^\Delta(t)y_i^\Delta(t).
\end{aligned}
$$

Since we are assuming y_d is a particular solution of the inhomogeneous equation (1.5), we must have

$$
\begin{aligned}
r(t) &= y_d^{\Delta\Delta}(t) + p(t)y_d^\Delta(t) + q(t)y_d(t) \\
&= v(t)\left(y_i^{\Delta\Delta}(t) + p(t)y_i^\Delta(t) + q(t)y_i(t)\right) + p(t)y_i(t)y_i^\sigma(t)v^\Delta(t)/y_i(t) \\
&\quad + y_i^{\Delta\sigma}(t)v^\Delta(t) + \left[y_i(t)y_i^\sigma(t)v^\Delta(t)\right]^\Delta/y_i^\sigma(t) - v^\Delta(t)y_i^\Delta(t).
\end{aligned}
$$

Now y_i is a solution of the homogeneous equation (3.2), so after simplifying we multiply by y_i^σ to get

$$
r(t)y_i^\sigma(t) = y_i^\sigma(t)p(t)\left[y_i(t)y_i^\sigma(t)v^\Delta(t)\right]/y_i(t) + \left[y_i(t)y_i^\sigma(t)v^\Delta(t)\right]^\Delta + y_i^\sigma(t)v^\Delta(t)\left(y_i^{\Delta\sigma}(t) - y_i^\Delta(t)\right).
$$

Make the substitution $u = y_iy_i^\sigma v^\Delta$, then use the simple formula $f^\sigma - f = \mu f^\Delta$ to get

$$
\begin{aligned}
r(t)y_i^\sigma(t) &= y_i^\sigma(t)p(t)u(t)/y_i(t) + u^\Delta(t) + y_i^\sigma(t)v^\Delta(t)\mu(t)y_i^{\Delta\Delta}(t) \\
&= u^\Delta(t) + \frac{p(t)y_i^\sigma(t) + \mu(t)y_i^{\Delta\Delta}(t)}{y_i(t)}u(t).
\end{aligned}
$$

Using the simple formula $f^\sigma - f = \mu f^\Delta$ again and rearranging, we see that

$$
u^\Delta(t) + (p(t) - \mu(t)q(t))u(t) = r(t)y_i^\sigma(t).
$$

Focusing on the coefficient of u, we note that

$$
\begin{aligned}
p(t) - \mu(t)q(t) &= -(-p(t) + \mu(t)q(t)) \\
&= -\ominus\left(\ominus(-p(t) + \mu(t)q(t))\right) \\
&= \frac{\ominus(-p(t) + \mu(t)q(t))e_{\ominus(-p+\mu q)}(t, a)}{[1 + \mu(t)\left(\ominus(-p(t) + \mu(t)q(t))\right)]e_{\ominus(-p+\mu q)}(t, a)} \\
&= \frac{e_{\ominus(-p+\mu q)}^\Delta(t, a)}{e_{\ominus(-p+\mu q)}^\sigma(t, a)}.
\end{aligned}
$$

Consequently we have that

$$
\left(e_{\ominus(-p+\mu q)}(t, a)u(t)\right)^\Delta = r(t)y_i^\sigma(t)e_{\ominus(-p+\mu q)}^\sigma(t, a).
$$

Solving while recalling that $u = y_i y_i^\sigma v^\Delta$, we arrive at

$$v(t) = \int \frac{e_{(-p+\mu q)}(t,a) \int r(t)y_i^\sigma(t)e_{\ominus(-p+\mu q)}(\sigma(t),a)\Delta t}{y_i(t)y_i^\sigma(t)} \Delta t,$$

so that via $y_d = y_i v$ we obtain (3.5). □

Remark 3.2. The regressivity assumption in (3.1) is not at all unusual, as it is automatic in the case $\mathbb{T} = \mathbb{R}$ since $\mu \equiv 0$, and it is assumed in the variation of parameters theorem on general time scales; see [3, Definition 3.3]. The following theorem is a simple corollary of Theorem 3.1.

Theorem 3.3 (Reduction of Order). *Assume p and q are real–valued right–dense continuous scalar functions of $t \in \mathbb{T}$ satisfying the regressivity condition*

$$(3.6) \qquad\qquad 1 + \mu(t)\left[-p(t) + \mu(t)q(t)\right] \neq 0, \quad t \in \mathbb{T}^\kappa.$$

If y_1 is a solution of the linear homogeneous second–order ordinary dynamic equation (2.5), then

$$(3.7) \qquad\qquad y_2(t) = y_1(t) \int \frac{e_{(-p+\mu q)}(t,a)}{y_1(t)y_1^\sigma(t)} \Delta t$$

is a second linearly independent solution of (2.5). Similarly, if y_2 is a solution of (2.5), then

$$(3.8) \qquad\qquad y_1(t) = y_2(t) \int \frac{e_{(-p+\mu q)}(t,a)}{y_2(t)y_2^\sigma(t)} \Delta t$$

is a second linearly independent solution of (2.5).

Proof. We will prove (3.7), since the proof of (3.8) is similar. Thus, assume y_1 is a solution of (2.5). Since $r(t) \equiv 0$ in this case, and general antiderivatives are used in Theorem 3.1, we may choose the constant of integration in (3.5) in such a way that

$$\int r(t)y_1^\sigma(t)e_{\ominus(-p+\mu q)}(\sigma(t),a)\Delta t = 1,$$

so that (3.5) becomes

$$y_d(t) = y_1(t) \int \frac{e_{(-p+\mu q)}(t,a)}{y_1(t)y_1^\sigma(t)} \Delta t = y_2(t),$$

in other words a particular solution of (2.5). One could also verify (3.7) directly using the delta derivative rules. To show linear independence, we calculate the Wronskian of y_1 and y_2, namely

$$(3.9) \qquad\qquad W(y_1, y_2)(t) = y_1(t)y_2^\Delta(t) - y_1^\Delta(t)y_2(t) = e_{(-p+\mu q)}(t,a) \neq 0$$

for all $t \in \mathbb{T}$ by the regressivity assumption (3.6); see [3, Theorem 2.48]. □

Remark 3.4. It is also possible to show the equivalence of the result shown in (3.5) with that due to the variation of parameters formula in equation (1.6). Let y_1 be a solution of the linear homogeneous equation (2.5). Assuming

(3.6), and appropriating the Wronskian of y_1 and the form of y_2 given in (3.7) as calculated in (3.9), the solution given in (1.6) is

$$
\begin{aligned}
y_d(t) \;=\; & y_2(t)\int_a^t \frac{y_1^\sigma(s)r(s)}{e_{(-p+\mu q)}(\sigma(s),a)}\Delta s - y_1(t)\int_a^t \frac{y_2^\sigma(s)r(s)}{e_{(-p+\mu q)}(\sigma(s),a)}\Delta s \\
\;=\; & \left(y_1(t)\int_a^t \frac{e_{(-p+\mu q)}(s,a)}{y_1(s)y_1^\sigma(s)}\Delta s\right)\int_a^t r(s)y_1^\sigma(s)e_{\ominus(-p+\mu q)}(\sigma(s),a)\Delta s \\
& -y_1(t)\int_a^t r(s)y_1^\sigma(s)e_{\ominus(-p+\mu q)}(\sigma(s),a)\int_a^{\sigma(s)}\frac{e_{(-p+\mu q)}(\xi,a)}{y_1(\xi)y_1^\sigma(\xi)}\Delta\xi\Delta s.
\end{aligned}
$$

(3.10)

If we use the integration by parts formula $\int fg^\Delta \Delta t = f(t)g(t) - \int f^\Delta g^\sigma \Delta t$, see [3, Theorem 1.77(vi)], on the integral in (3.5), where we have taken

$$
f(t) = \int_a^t r(s)y_1^\sigma(s)e_{\ominus(-p+\mu q)}(\sigma(s),a)\Delta s \quad\text{and}\quad g^\Delta(t) = \frac{e_{(-p+\mu q)}(t,a)}{y_1(t)y_1^\sigma(t)},
$$

we get (3.10).

The next corollary concerns another possible second–order linear dynamic equation discussed by Bohner and Peterson [3, (3.6)].

Corollary 3.5. *Let α, β and r be real–valued right–dense continuous scalar functions on $\mathbb{T}$ with α satisfying the regressivity condition $1 + \mu(t)\alpha(t) \neq 0$ for $t \in \mathbb{T}^\kappa$. For all $t \in \mathbb{T}$, let y_1 and y_2 satisfy*

(3.11)
$$
y_i^{\Delta\Delta}(t) + \alpha(t)y_i^{\Delta\sigma}(t) + \beta(t)y_i^\sigma(t) = 0, \quad i = 1, 2,
$$

with $W(t) := y_1(t)y_2^\Delta(t) - y_2(t)y_1^\Delta(t) \neq 0$. The general solution of the linear inhomogeneous second–order ordinary dynamic equation

(3.12)
$$
y^{\Delta\Delta}(t) + \alpha(t)y^{\Delta\sigma}(t) + \beta(t)y^\sigma(t) = r(t), \quad t \in \mathbb{T}^{\kappa^2},
$$

is

(3.13)
$$
y(t) = c_1 y_1(t) + c_2 y_2(t) + y_1(t)\int \frac{e_{\ominus\alpha}(t,a)\int r(t)y_1^\sigma(t)e_\alpha(\sigma(t),a)\Delta t}{y_1(t)y_1^\sigma(t)}\Delta t
$$

or

(3.14)
$$
y(t) = c_1 y_1(t) + c_2 y_2(t) + y_2(t)\int \frac{e_{\ominus\alpha}(t,a)\int r(t)y_2^\sigma(t)e_\alpha(\sigma(t),a)\Delta t}{y_2(t)y_2^\sigma(t)}\Delta t,
$$

where c_1 and c_2 are arbitrary constants, and where $e_\alpha(\cdot,a)$ is the time-scale exponential function.

Proof. Rewriting (3.12) using the simple formula $f^\sigma = f + \mu f^\Delta$, we see that we arrive at an equation of the form (1.5), where

$$
p(t) = \frac{\alpha(t) + \mu(t)\beta(t)}{1 + \mu(t)\alpha(t)} \quad\text{and}\quad q(t) = \frac{\beta(t)}{1 + \mu(t)\alpha(t)}.
$$

Then we see that the term $-p + \mu q$ in Theorem 3.1 above is given by

$$
-p(t) + \mu(t)q(t) = \frac{-\alpha(t)}{1 + \mu(t)\alpha(t)} = \ominus\alpha(t),
$$

and this corollary follows.   □

4. SECOND–ORDER LINEAR ORDINARY DIFFERENCE EQUATIONS

In this section we recount the results of the previous section when $\mathbb{T} = \mathbb{Z}$, that is to say for second–order linear ordinary difference equations. The following corollaries to Theorem 3.1 are obtained by simply taking $\mathbb{T} = \mathbb{Z}$ in Theorem 3.1 above. The first uses the forward difference operator form of the second–order linear equation (1.2), and the second uses the shift form (1.3).

Theorem 4.1. *Let p, q and r be real–valued scalar functions of $t \in \mathbb{Z}$ with p and q satisfying the regressivity condition*

$$(4.1) \qquad\qquad\qquad\qquad 1 - p(t) + q(t) \neq 0, \quad t \in \mathbb{Z}.$$

Let y_1 and y_2 satisfy

$$\Delta^2 y_i(t) + p(t)\Delta y_i(t) + q(t)y_i(t) = 0, \quad i = 1, 2,$$

and $W(t) := y_1(t)\Delta y_2(t) - y_2(t)\Delta y_1(t) \neq 0$. The general solution of the linear inhomogeneous second–order ordinary difference equation

$$\Delta^2 y(t) + p(t)\Delta y(t) + q(t)y(t) = r(t), \quad t \in \mathbb{Z},$$

is given by

$$(4.2) \qquad y(t) = c_1 y_1(t) + c_2 y_2(t) + y_1(t) \sum \frac{\prod_{j=a}^{t-1}(1 - p(j) + q(j)) \sum \frac{r(t)y_1(t+1)}{\prod_{j=a}^{t}(1-p(j)+q(j))}}{y_1(t)y_1(t+1)}$$

or

$$(4.3) \qquad y(t) = c_1 y_1(t) + c_2 y_2(t) + y_2(t) \sum \frac{\prod_{j=a}^{t-1}(1 - p(j) + q(j)) \sum \frac{r(t)y_2(t+1)}{\prod_{j=a}^{t}(1-p(j)+q(j))}}{y_2(t)y_2(t+1)},$$

and where c_1 and c_2 are arbitrary constants.

Theorem 4.2. *Let α, β and r be real–valued scalar functions of $t \in \mathbb{Z}$ with regressivity condition $\beta(t) \neq 0$ for $t \in \mathbb{Z}$. Let y_1 and y_2 satisfy*

$$y_i(t + 2) + \alpha(t)y_i(t + 1) + \beta(t)y_i(t) = 0, \quad i = 1, 2,$$

with $W(t) := y_1(t)y_2(t + 1) - y_2(t)y_1(t + 1) \neq 0$. The general solution of the linear inhomogeneous second–order ordinary difference equation

$$(4.4) \qquad\qquad y(t + 2) + \alpha(t)y(t + 1) + \beta(t)y(t) = r(t), \quad t \in \mathbb{Z},$$

is given by

$$(4.5) \qquad y(t) = c_1 y_1(t) + c_2 y_2(t) + y_1(t) \sum \frac{\prod_{j=a}^{t-1}\beta(j) \sum \frac{r(t)y_1(t+1)}{\prod_{j=a}^{t}\beta(j)}}{y_1(t)y_1(t+1)}$$

or

$$(4.6) \qquad y(t) = c_1 y_1(t) + c_2 y_2(t) + y_2(t) \sum \frac{\prod_{j=a}^{t-1}\beta(j) \sum \frac{r(t)y_2(t+1)}{\prod_{j=a}^{t}\beta(j)}}{y_2(t)y_2(t+1)},$$

where c_1 and c_2 are arbitrary constants.

In [11, Section 3] the authors applied their alternative formula (1.7) to give a general treatment of (1.1) in the case of constant coefficient functions $p(t) \equiv 2P$ and $q(t) \equiv Q$ with P and Q real constants with $Q \neq 0$ and $P^2 \neq Q$. We follow suit by giving a general treatment of (4.4) with constant coefficients using the results above from Theorem 4.2. To wit, we analyze the ordinary difference equation

$$(4.7) \qquad\qquad y(t+2) + 2\alpha y(t+1) + \beta y(t) = r(t), \quad t \in \mathbb{Z},$$

where α and β are real constants with $\beta \neq 0$ and $\alpha^2 \neq \beta$. The solution of (4.7) can be written as

$$y(t) = c_1 \left(-\alpha - \sqrt{\alpha^2 - \beta}\right)^t + c_2 \left(-\alpha + \sqrt{\alpha^2 - \beta}\right)^t + y_d.$$

From Theorem 4.2 we know that a particular solution to the inhomogeneous equation (4.7) has the form

$$y_d(t) = y_i(t) \sum_{x=a}^{t-1} \frac{\prod_{j=a}^{x-1} \beta \sum_{s=a}^{x-1} \frac{r(s)y_i(s+1)}{\prod_{j=a}^{s} \beta}}{y_i(x)y_i(x+1)}, \quad i = 1, 2,$$

which simplifies to

$$y_d(t) = \sum_{x=a}^{t-1} \sum_{s=a}^{x-1} r(s) \left(-\alpha \pm \sqrt{\alpha^2 - \beta}\right)^{t+s-2x} \beta^{x-1-s}.$$

Thus, complete expressions for the solution of (4.7) are given by

$$y(t) = c_1 \lambda_1^t + c_2 \lambda_2^t + \sum_{x=a}^{t-1} \sum_{s=a}^{x-1} r(s) \lambda_1^{t+s-2x} \beta^{x-1-s}$$

or

$$y(t) = c_1 \lambda_1^t + c_2 \lambda_2^t + \sum_{x=a}^{t-1} \sum_{s=a}^{x-1} r(s) \lambda_2^{t+s-2x} \beta^{x-1-s},$$

where for simplicity we have taken $\lambda_1 = -\alpha - \sqrt{\alpha^2 - \beta}$ and $\lambda_2 = -\alpha + \sqrt{\alpha^2 - \beta}$.

Remark 4.3. A general treatment of (1.5) with constant coefficients on arbitrary time scales and even quantum equations is made difficult by the fact that, as seen in (3.5), even with p and q taken to be constant functions, the term $-p + \mu q$ is not constant except for the very special cases of $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$.

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